

Strongly Complete Axiomatizations for Logics of Dynamical Systems

Somayeh Chopoghloo

Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

*Seminari Cuc, Department of Philosophy, University of Barcelona
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Dynamical Systems

Definition

Let X be a non-empty set and ϕ an action of a monoid T on X . A **dynamical system over T** is a pair $\langle X, f^t \rangle_{t \in T}$ where for each $t \in T$, $f^t : X \rightarrow X$ is a function defined by $f^t(x) = \phi(t, x)$.

- Dynamical systems over the monoid $(\omega, +)$ are called **discrete-time** systems. They are typically represented by pairs $\langle X, f \rangle$, where $f : X \rightarrow X$ is a map.
- The state space X is often equipped with additional structure (e.g., an order, topology, metric, or measure) that f preserves.

Motivation

Question

Which logical theories are appropriate for describing and reasoning about dynamical systems?

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Which logical theories are appropriate for describing and reasoning about dynamical systems?

- Dynamical systems can be formalized within standard foundations such as [set theory](#) or [second-order arithmetic](#), but these are generally undecidable.
- For many applications, especially in [automated theorem proving](#), a complete or decidable logic is preferable.
- [One approach is to reason about dynamical systems in a framework that cannot directly formalize arithmetic, and modal logics being particularly well-suited for this.](#)

D. Fernández-Duque (2017). *Logical Dynamics*. Paul Sabatier University, Toulouse, France.

Mathematical Structures

- A **dynamic topological system** is a triple $\langle X, \tau, f \rangle$, where $\langle X, \tau \rangle$ is a topological space and $f : X \rightarrow X$ is a continuous map (in particular, a homeomorphism when invertible).
- A **dynamic Kripke frame** (or **expanding frame**) is a triple $\langle W, R, f \rangle$, where $\langle W, R \rangle$ is a preorder (respectively, a poset) and $f : W \rightarrow W$ is R -monotone, i.e., wRv implies $f(w)Rf(v)$.
- A **persistent frame** is an expanding frame in which f satisfies the following condition:

if $f(w)Ru$, then there exists v such that wRv and $f(v) = u$.

- There is a one-to-one correspondence between dynamic Kripke frames and dynamic topological systems over Alexandrov spaces.

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Syntax

The language \mathcal{L}_{DTL} is inductively defined by the following grammar:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \blacksquare\varphi \mid \bigcirc\varphi \mid \square\varphi$$

- The topological modality \blacksquare is read as ‘interior’, and the temporal modalities \bigcirc and \square are read ‘next’ and ‘henceforth’, respectively.
- The modalities \blacklozenge (‘closure’) and \lozenge (‘eventually’) are defined as abbreviations:

$$\blacklozenge\varphi := \neg\blacksquare\neg\varphi, \quad \lozenge\varphi := \neg\square\neg\varphi.$$

Semantics

A **dynamic topological model**, $\mathfrak{M} = \langle X, \tau, f, \nu \rangle$, is a dynamic topological system that is equipped with a valuation $\nu : \mathbb{P} \rightarrow \mathcal{P}(X)$.

The satisfiability relation for the new formulas is defined inductively as

$$\mathfrak{M}, x \models \blacksquare \varphi \text{ iff } x \in \mathbb{I}\{y \in X \mid \mathfrak{M}, y \models \varphi\};$$

$$\mathfrak{M}, x \models \bigcirc \varphi \text{ iff } \mathfrak{M}, f(x) \models \varphi;$$

$$\mathfrak{M}, x \models \square \varphi \text{ iff } \mathfrak{M}, f^n(x) \models \varphi \text{ for all } n \in \omega.$$

Validity

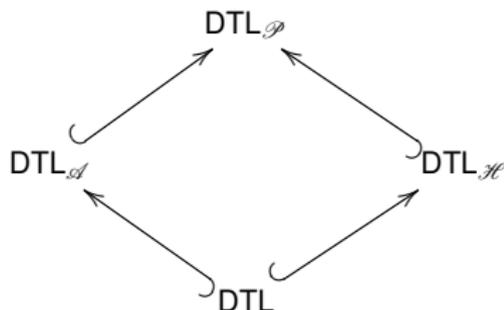
- A formula φ is **valid in a model** \mathfrak{M} , written $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, x \models \varphi$ for all $x \in X$.
- A formula φ is **valid in a system** \mathfrak{F} , written $\mathfrak{F} \models \varphi$, if φ is valid in every model based on \mathfrak{F} .
- A formula φ is **valid in a class** \mathcal{C} of systems, written $\models_{\mathcal{C}} \varphi$, if φ is valid in every $\mathfrak{F} \in \mathcal{C}$.
- A formula φ is a **semantic consequence** of a set Γ over \mathcal{C} , written $\Gamma \models_{\mathcal{C}} \varphi$, if for every model \mathfrak{M} based on some $\mathfrak{F} \in \mathcal{C}$ and every $x \in X$,
if $\mathfrak{M}, x \models \gamma$ for all $\gamma \in \Gamma$, then $\mathfrak{M}, x \models \varphi$.

DTL

Definition (Kremer & Mints 2005)

The set of all formulas in \mathcal{L}_{DTL} valid over the class of all dynamic topological systems is called **dynamic topological logic (DTL)**.

- More generally, $\text{DTL}_{\mathcal{C}}$ denotes the set of all formulas in \mathcal{L}_{DTL} that are valid over a class \mathcal{C} of dynamic topological systems.
- The following inclusions among dynamic topological logics hold:



P. Kremer and G. Mints (2005). Dynamic topological logic. *Annals of Pure and Applied Logic*, 131, 133–158.

Some Known Facts

- The classes of **dynamical systems over Alexandrov spaces**, **Poincaré recurrent systems**, and **minimal systems** are definable.
- DTL is **undecidable**. When restricted to invertible systems, the set of valid formulas ($\text{DTL}_{\mathcal{H}}$) is **not recursively axiomatizable** (Konev et al., 2006).
- DTL is **recursively enumerable** but **not finitely axiomatizable** (Fernández-Duque, 2009, 2014).

B. Konev, R. Kontchakov, F. Wolter, and M. Zakharyashev (2006). Dynamic topological logics over spaces with continuous functions. *Advances in Modal Logic*, 6, 299–318.

D. Fernández-Duque (2009). Non-deterministic semantics for dynamic topological logic. *Annals of Pure and Applied Logic*, 157(2-3), 110–121.

D. Fernández-Duque (2014). Non-finite axiomatizability of dynamic topological logic. *ACM Transactions on Computational Logic*, 15(1), 4.

Main properties

Consider $\Sigma = \{\neg\Box p\} \cup \{p, \bigcirc p, \bigcirc^2 p, \dots\}$. For any finite subset Σ' of Σ , there is $n \in \omega$ such that

$$\Sigma' \subseteq \{\neg\Box p\} \cup \{p, \bigcirc p, \dots, \bigcirc^n p\}.$$

For such n , take the model $\mathfrak{M}_n = \langle \mathcal{X}, f, V \rangle$ as follows:

$$X = \{x_i \mid 0 \leq i \leq n+1\};$$

τ is the discrete topology on X ;

$$f(x_i) = x_{i+1} \text{ for } 0 \leq i \leq n;$$

$$V(p) = \{x_i \mid 0 \leq i \leq n\}.$$

In this case, $\mathfrak{M}_n, x_0 \models \Sigma'$. So, DTL is **semantically non-compact**.

- No finitary proof system for DTL could be strongly complete.

Main properties

Let $\Gamma = \{\bigcirc^n p \mid n \in \omega\}$ and $\varphi = \Box p$. Then, $\Gamma \vDash_{\mathcal{J}} \varphi$. But, $\blacksquare \Gamma \not\vDash_{\mathcal{J}} \blacksquare \varphi$. To see this, consider the model $\mathfrak{M} = \langle \mathbb{R}, f, V \rangle$ where $f(x) = 2x$ and $v(p) = (-1, 1)$. In this case, $\mathfrak{M}, 0 \vDash \blacksquare \gamma$ for all $\gamma \in \Gamma$, but $\mathfrak{M}, 0 \not\vDash \blacksquare \Box p$.

- So, the strong necessitation rule for \blacksquare , i.e.

if $\Gamma \vDash_{\mathcal{J}} \varphi$ then $\blacksquare \Gamma \vDash_{\mathcal{J}} \blacksquare \varphi$,

when Γ is an infinite set, is not valid in the class \mathcal{J} .

The Proof System T_{DTL}^ω

Consider the following axiom schemes:

Taut	All propositional tautologies
K_{\blacksquare}	$\blacksquare(\varphi \rightarrow \psi) \rightarrow (\blacksquare\varphi \rightarrow \blacksquare\psi)$
T_{\blacksquare}	$\blacksquare\varphi \rightarrow \varphi$
4_{\blacksquare}	$\blacksquare\varphi \rightarrow \blacksquare\blacksquare\varphi$
Func_{\bigcirc}	$\bigcirc\neg\varphi \leftrightarrow \neg\bigcirc\varphi$
FP_{\square}	$\square\varphi \rightarrow \varphi \wedge \bigcirc\square\varphi$
Cont	$\bigcirc\blacksquare\varphi \rightarrow \blacksquare\bigcirc\varphi$

The Proof System T_{DTL}^ω

The abstract derivation relation $\Gamma \vdash_{T^\omega} \varphi$ is defined as the smallest relation that is closed under the following rules:

Ax	If φ is an axiom then $\vdash_{T^\omega} \varphi$
Id	$\varphi \vdash_{T^\omega} \varphi$
MP	$\varphi, \varphi \rightarrow \psi \vdash_{T^\omega} \psi$
Nec \blacksquare	If $\vdash_{T^\omega} \varphi$ then $\vdash_{T^\omega} \blacksquare\varphi$
SNec \bigcirc	If $\Gamma \vdash_{T^\omega} \varphi$ then $\bigcirc\Gamma \vdash_{T^\omega} \bigcirc\varphi$
Ω -Ind	$\{\bigcirc^n\varphi \mid n \in \omega\} \vdash_{T^\omega} \Box\varphi$
Wek	If $\Gamma \vdash_{T^\omega} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{T^\omega} \varphi$
Ded	If $\Gamma, \varphi \vdash_{T^\omega} \psi$ then $\Gamma \vdash_{T^\omega} \varphi \rightarrow \psi$
Cut	If $\Gamma \vdash_{T^\omega} \Delta$ and $\Delta \vdash_{T^\omega} \varphi$, then $\Gamma \vdash_{T^\omega} \varphi$

Some Results

Theorem (Ch & Moniri 2022)

- The system T_{DTL}^{ω} is strongly sound and complete for the class \mathcal{T} of all dynamic topological systems; that is, $\Gamma \vdash_{T^{\omega}} \varphi$ iff $\Gamma \vDash_{\mathcal{T}} \varphi$.
- The extended system $T_{\text{DTL}_{\mathcal{A}}}^{\omega} = T_{\text{DTL}}^{\omega} + \text{if } \Gamma \vdash \varphi \text{ then } \blacksquare \Gamma \vdash \blacksquare \varphi$ is strongly sound and complete for the class \mathcal{A} of all dynamic Alexandrov systems.

Some Results

Theorem (Ch & Moniri 2022)

- The system T_{DTL}^ω is strongly sound and complete for the class \mathcal{F} of all dynamic topological systems; that is, $\Gamma \vdash_{T^\omega} \varphi$ iff $\Gamma \vDash_{\mathcal{F}} \varphi$.
- The extended system $T_{DTL_{\mathcal{A}}}^\omega = T_{DTL}^\omega + \text{if } \Gamma \vdash \varphi \text{ then } \blacksquare \Gamma \vdash \blacksquare \varphi$ is strongly sound and complete for the class \mathcal{A} of all dynamic Alexandrov systems.

Theorem

- The logic $DTL_{\mathcal{H}}$ (with homeomorphisms) is axiomatized by extending T_{DTL}^ω with the axiom scheme $\blacksquare \bigcirc \varphi \rightarrow \bigcirc \blacksquare \varphi$.
- The logic $DTL_{\mathcal{P}}$ (persistent frames) is axiomatized by $T_{DTL_{\mathcal{A}}}^\omega + \blacksquare \bigcirc \varphi \rightarrow \bigcirc \blacksquare \varphi$.

S. Chopoghloo and M. Moniri (2022). An infinitary axiomatization of dynamic topological logic. *Logic Journal of the IGPL*, 30(1), 124–142.

Strong Completeness of T_{DTL}^{ω}

Sketch of proof

Definition

We define the set A^{ω} as the smallest subset of $\vdash_{T_{DTL}^{\omega}}$ that contains all instances of the rule Ω -Ind and is closed under the rules $SNec_{\blacksquare}$, $SNec_{\bigcirc}$ and Imp.

Strong Completeness of T_{DTL}^ω

Sketch of proof

Definition

We define the set A^ω as the smallest subset of $\vdash_{T_{DTL}^\omega}$ that contains all instances of the rule Ω -Ind and is closed under the rules $SNec_{\blacksquare}$, $SNec_{\bigcirc}$ and Imp.

Lemma

A^ω is a countable set.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Definition

A set of formulas Γ is called $T_{\mathcal{A}}^{\omega}$ -saturated iff

1. Γ is finitely $T_{\mathcal{A}}^{\omega}$ -consistent, i.e. every finite subset of Γ is $T_{\mathcal{A}}^{\omega}$ -consistent,
2. Γ is negation-complete, i.e. for all φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, and
3. Γ is A^{ω} -closed, i.e. if $(\Sigma, \varphi) \in A^{\omega}$ and $\neg\varphi \in \Gamma$, then there exists $\sigma \in \Sigma$ such that $\neg\sigma \in \Gamma$.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Let Γ be a $T_{\mathcal{A}}^{\omega}$ -saturated set. Then,

1. if $\blacklozenge\varphi \in \Gamma$, then φ is $T_{\mathcal{A}}^{\omega}$ -consistent;
2. if $\blacklozenge\varphi \in \Gamma$ and $\blacksquare\psi \in \Gamma$, then $\blacklozenge(\varphi \wedge \psi) \in \Gamma$;
3. if $\blacklozenge\varphi \in \Gamma$, then for any formula ψ , either $\blacklozenge(\varphi \wedge \psi) \in \Gamma$ or $\blacklozenge(\varphi \wedge \neg\psi) \in \Gamma$;
4. if $\blacklozenge(\varphi \wedge \neg\psi) \in \Gamma$ and $(\Sigma, \psi) \in A^{\omega}$, then there is $\sigma \in \Sigma$ such that $\blacklozenge(\varphi \wedge \neg\psi \wedge \neg\sigma) \in \Gamma$;
5. $\Box\varphi \in \Gamma$ iff $\{\bigcirc^n\varphi \mid n \in \omega\} \subseteq \Gamma$.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Lemma

Suppose that Γ is a $T_{\mathcal{A}}^{\omega}$ -consistent set and (Σ, φ) is an element of A^{ω} . Then there exists $\sigma \in \Sigma$ such that $\Gamma \cup \{\sigma \rightarrow \varphi\}$ is $T_{\mathcal{A}}^{\omega}$ -consistent.

Lindenbaum's Lemma

Suppose that Γ is a $T_{\mathcal{A}}^{\omega}$ -consistent set. Then there exists a $T_{\mathcal{A}}^{\omega}$ -saturated set Γ^* such that $\Gamma \subseteq \Gamma^*$.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Definition (Canonical Kripke Model)

The canonical Kripke model is the tuple $\mathfrak{M}_c = \langle W_c, R_c, f_c, V_c \rangle$ where

- W_c is the set of all $T_{\mathcal{A}}^{\omega}$ -saturated sets;
- R_c is the binary relation on W_c that is defined by $\Gamma R_c \Delta$ iff for all formula φ , $\blacksquare\varphi \in \Gamma$ implies $\varphi \in \Delta$;
- f_c is the function defined by $f_c(\Gamma) = \{\varphi \mid \bigcirc\varphi \in \Gamma\}$ for any Γ in W_c ;
- V_c is the valuation defined by $V_c(p) = \{\Gamma \in W_c \mid p \in \Gamma\}$ for any $p \in \mathbb{P}$.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Definition (Canonical Kripke Model)

The canonical Kripke model is the tuple $\mathfrak{M}_c = \langle W_c, R_c, f_c, V_c \rangle$ where

- W_c is the set of all $T_{\mathcal{A}}^{\omega}$ -saturated sets;
- R_c is the binary relation on W_c that is defined by $\Gamma R_c \Delta$ iff for all formula φ , $\blacksquare\varphi \in \Gamma$ implies $\varphi \in \Delta$;
- f_c is the function defined by $f_c(\Gamma) = \{\varphi \mid \bigcirc\varphi \in \Gamma\}$ for any Γ in W_c ;
- V_c is the valuation defined by $V_c(p) = \{\Gamma \in W_c \mid p \in \Gamma\}$ for any $p \in \mathbb{P}$.

Lemma

\mathfrak{M}_c is a dynamic Kripke model.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Lemma

Consider the model \mathfrak{M}_c . Suppose that Γ is a $T_{\mathcal{A}}^{\omega}$ -saturated set and φ is a formula. Then, if for all $\Delta \in W_c$, $\Gamma R_c \Delta$ implies $\varphi \in \Delta$, then $\blacksquare\varphi \in \Gamma$.

- The usual way is to show that the set

$$\Delta^- = \{\psi \mid \blacksquare\psi \in \Gamma\} \cup \{\neg\varphi\}$$

is $T_{\mathcal{A}}^{\omega}$ -consistent. But, it does not work here, since Γ is only finitely $T_{\mathcal{A}}^{\omega}$ -consistent.

Strong Completeness of T_{DTL}^ω

Sketch of proof

Proof.

Suppose that $\blacksquare\varphi \notin \Gamma$. Then, $\blacklozenge\neg\varphi \in \Gamma$. We show that there is $\Delta^* \in W_c$ such that $\Gamma R_c \Delta^*$ and $\varphi \notin \Delta^*$. To this purpose, suppose that $\varphi_0, \varphi_1, \dots$ is an enumeration of all formulas of \mathcal{L}_{DTL} , and $(\Sigma_0, \psi_0), (\Sigma_1, \psi_1), \dots$ is an enumeration of all members of A^ω .

We inductively define a sequence $\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_k \subseteq \dots$ of finite sets with this property that $\blacklozenge(\bigwedge \Delta_k) \in \Gamma$ as follows:

Step 0 Let $\Delta_0 := \{\neg\varphi\}$.

Step k+1 Let Δ_k be given such that $\blacklozenge(\bigwedge \Delta_k) \in \Gamma$. We define ...

Finally, we define

$$\Delta^* := \bigcup_{k \in \omega} \Delta_k.$$



Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Truth Lemma

Consider the canonical Kripke model \mathfrak{M}_c . Then for any formula φ of \mathcal{L}_{DTL} and Γ in W_c , we have

$$\mathfrak{M}_c, \Gamma \models \varphi \text{ iff } \varphi \in \Gamma.$$

Strong Completeness Theorem

If $\Gamma \models_{\mathcal{A}} \varphi$ then $\Gamma \vdash_{T_{\mathcal{A}}^{\omega}} \varphi$.

Strong Completeness of $T_{DTL_{\mathcal{A}}}^{\omega}$

Sketch of proof

Truth Lemma

Consider the canonical Kripke model \mathfrak{M}_c . Then for any formula φ of \mathcal{L}_{DTL} and Γ in W_c , we have

$$\mathfrak{M}_c, \Gamma \models \varphi \text{ iff } \varphi \in \Gamma.$$

Strong Completeness Theorem

If $\Gamma \models_{\mathcal{A}} \varphi$ then $\Gamma \vdash_{T_{\mathcal{A}}^{\omega}} \varphi$.

Corollary

A set of formulas is $T_{\mathcal{A}}^{\omega}$ -saturated iff it is maximal $T_{\mathcal{A}}^{\omega}$ -consistent.

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Syntax

The language \mathcal{L}_{ITL} is inductively defined by the following grammar:

$$\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \bigcirc \varphi \mid \diamond \varphi \mid \square \varphi$$

Topological Semantics

A **intuitionistic topological model** $\mathfrak{M} = \langle X, \tau, f, v \rangle$, is a dynamic topological system that is equipped with a valuation function $v : \mathbb{P} \rightarrow \tau$.

$$v(\varphi \rightarrow \psi) = \mathbb{I}((X \setminus v(\varphi)) \cup v(\psi))$$
$$v(\Box \varphi) = \mathbb{I}\left(\bigcap_{n \in \omega} f^{-n}(v(\varphi))\right).$$

Definition (Fernández-Duque 2018)

The **intuitionistic temporal logic of continuous functions** (ITL^c) is the set of all formulas in \mathcal{L}_{ITL} valid over the class of all dynamic topological systems.

D. Fernández-Duque (2018). The intuitionistic temporal logic of dynamical systems. *Logical Methods in Computer Science*, 14, 1–35.

Kripke Semantics

An **intuitionistic Kripke model** is a tuple $\mathfrak{M} = \langle W, \preceq, f, v \rangle$ where

- $\langle W, \preceq \rangle$ is a partially ordered set,
- $f : W \rightarrow W$ is a \preceq -monotone function, and
- $v : \mathbb{P} \rightarrow \mathcal{P}(W)$ is a valuation that is upward-closed under \preceq , i.e. if $w \in v(p)$ and $w \preceq v$, then $v \in v(p)$.

The satisfiability relation for the new formulas is defined inductively as

$\mathfrak{M}, w \vDash \varphi \rightarrow \psi$ iff for all $v \succeq w$, if $\mathfrak{M}, v \vDash \varphi$ then $\mathfrak{M}, v \vDash \psi$;

Definition (Boudou et al. 2017)

- The **intuitionistic temporal logic of expanding frames** (ITL^e) is the set of all formulas in \mathcal{L}_{ITL} valid over the class of all dynamic Kripke (expanding) frames.
- The **intuitionistic temporal logic of persistent frames** (ITL^p) consists of all formulas valid in the class \mathcal{P} of persistent frames.

J. Boudou, M. Diéguez, and D. Fernández-Duque (2017). A decidable intuitionistic temporal logic. In *Proceedings of the 26th EACSL Annual Conference on Computer Science Logic (CSL)*, vol. 82, pp. 14:1–14:17.

Review of Results

- ITL^e has the **strong finite model property** and so **is decidable**, whereas ITL^p lacks the finite model property. (Boudou et al. 2017)
- Fernández-Duque proved the decidability of a fragment of ITL^c with \bigcirc , \diamond , and the universal modality \forall , while Boudou et al. showed its completeness over various familiar topological spaces.
- Extensions of ITL^e and ITL^p with binary temporal operators have been studied in Balbiani et al. 2020.

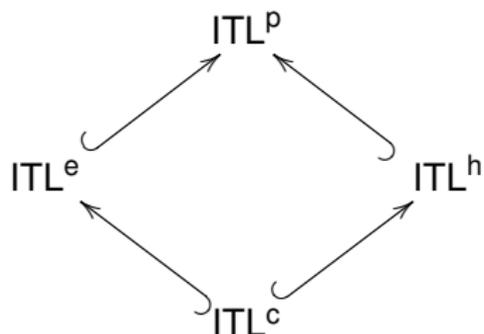
P. Balbiani, J. Boudou, M. Diéguez, and D. Fernández-Duque (2020). Intuitionistic linear temporal logics. *ACM Transactions on Computational Logic*, 21(2), 14:1–14:32.

J. Boudou, M. Diéguez, and D. Fernández-Duque (2022). Complete intuitionistic temporal logics for topological dynamics. *Journal of Symbolic Logic*, 87 (3), 995–1022.

J. Boudou, M. Diéguez, and D. Fernández-Duque (2017). A decidable intuitionistic temporal logic. In *Proceedings of the 26th EACSL Annual Conference on Computer Science Logic (CSL)*, vol. 82, pp. 14:1–14:17.

D. Fernández-Duque (2018). The intuitionistic temporal logic of dynamical systems. *Logical Methods in Computer Science*, 14, 1–35.

We have the following inclusions among intuitionistic temporal logics:



where ITL^h is intuitionistic temporal logic of homeomorphism.

Some Examples

Example 1

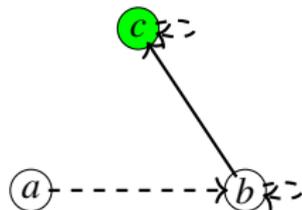
Consider the model $\mathfrak{M} = \langle W, \preceq, f, v \rangle$ as follows:

$$W = \{a, b, c\};$$

$\preceq =$ the reflexive closure of $\{(b, c)\}$; (In the following figure, the reflexive arrows are omitted.)

$$f = \{(a, b), (b, b), (c, c)\};$$

$$v(p) = \{c\} \text{ where } p \in \mathbb{P}.$$



In this case, $\mathfrak{M}, a \not\models \neg \Box \neg p \rightarrow \Diamond p$ and $\mathfrak{M}, a \not\models \neg \Diamond \neg p \rightarrow \Box p$.

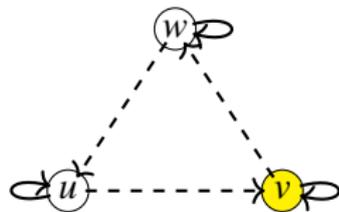
- Neither $\Diamond \phi \leftrightarrow \neg \Box \neg \phi$ nor $\Box \phi \leftrightarrow \neg \Diamond \neg \phi$ are valid intuitionistically.

Some Examples

Example 2

Let $\Gamma = \{\bigcirc^n p \mid n \in \omega\}$ and $\varphi = \Box p$ where $p \in \mathbb{P}$. Then, $\Gamma \vDash_{\mathcal{K}} \varphi$. But, $\diamond \Gamma \not\vDash_{\mathcal{K}} \diamond \varphi$. To see this, consider the model \mathfrak{M} as follows:

$W = \{u, v, w\}$;
 $\preceq =$ the identity relation;
 $f = \{(u, v), (v, w), (w, u)\}$;
 $v(p) = \{v\}$.



In this case, $\mathfrak{M}, u \vDash \bigcirc^n p$ for all $n \in \omega$, but $\mathfrak{M}, u \not\vDash \diamond \Box p$.

- The strong necessitation rule for \diamond , i.e.,
if $\Gamma \vDash_{\mathcal{K}} \varphi$ then $\diamond \Gamma \vDash_{\mathcal{K}} \diamond \varphi$,
when Γ is an infinite set of formulas, is not valid.

Some Examples

Example 3

Consider the infinite set $\Gamma = \{\neg\Box p\} \cup \{p, \bigcirc p, \bigcirc^2 p, \dots\}$. For every finite $\Gamma' \subseteq \Gamma$, there is $n \in \omega$ such that

$$\Gamma' \subseteq \{\neg\Box p\} \cup \{p, \bigcirc p, \dots, \bigcirc^n p\}.$$

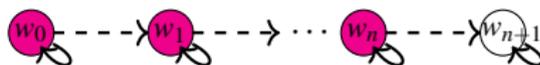
For such n , take the model \mathfrak{M}^n as follows:

$$W = \{w_j \mid 0 \leq j \leq n+1\};$$

$\preceq =$ the identity relation;

$$f(w_j) = w_{j+1} \text{ for } 0 \leq j \leq n;$$

$$v(p) = \{w_j \mid 0 \leq j \leq n\}.$$



It is easy to see that $\mathfrak{M}^n, w_0 \models \Gamma'$. However, there is no model which satisfies Γ .

- So, ITL^e is semantically non-compact.

The Proof System $T_{ITL^e}^\omega$

Consider the following axiom schemes:

InTaut	All intuitionistic tautologies
Con \bigcirc	$\bigcirc \perp \rightarrow \perp$
D \bigcirc	$\bigcirc(\varphi \vee \psi) \rightarrow \bigcirc\varphi \vee \bigcirc\psi$
Com \diamond	$\bigcirc\diamond\varphi \rightarrow \diamond\bigcirc\varphi$
FP \diamond	$\varphi \vee \bigcirc\diamond\varphi \rightarrow \diamond\varphi$
FP \square	$\square\varphi \rightarrow \varphi \wedge \bigcirc\square\varphi$

The Proof System $T_{ITL^e}^\omega$

The abstract derivation relation $\Gamma \vdash_{T_e^\omega} \varphi$ is defined as the smallest relation that is closed under the following rules:

Ax	If φ is an axiom then $\vdash_{T_e^\omega} \varphi$
Id	$\varphi \vdash_{T_e^\omega} \varphi$
MP	$\varphi, \varphi \rightarrow \psi \vdash_{T_e^\omega} \psi$
SNec \circ	If $\Gamma \vdash_{T_e^\omega} \varphi$ then $\circ\Gamma \vdash_{T_e^\omega} \circ\varphi$
Ω -Ind	$\{\circ^n \varphi \mid n \in \omega\} \vdash_{T_e^\omega} \square\varphi$
Ω -CoInd	$\{\circ^n \varphi \rightarrow \psi \mid n \in \omega\} \vdash_{T_e^\omega} \diamond\varphi \rightarrow \psi$
Dis	If $\Gamma \vdash_{T_e^\omega} \varphi$ then $\Gamma \vee \psi \vdash_{T_e^\omega} \varphi \vee \psi$
Wek	If $\Gamma \vdash_{T_e^\omega} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{T_e^\omega} \varphi$
Ded	If $\Gamma, \varphi \vdash_{T_e^\omega} \psi$ then $\Gamma \vdash_{T_e^\omega} \varphi \rightarrow \psi$
Cut	If $\Gamma \vdash_{T_e^\omega} \Delta$ and $\Delta \vdash_{T_e^\omega} \varphi$, then $\Gamma \vdash_{T_e^\omega} \varphi$

Strong Completeness of $T_{ITL^e}^\omega$

The sketch of proof

Definition

The set \mathcal{S}^ω is defined as the smallest subset of the derivation relation $\vdash_{T_e^\omega}$ that contains all instances of the rules Ω -Ind and Ω -CoInd and is closed under the rules $SNec_\circ$, Dis and Imp.

Strong Completeness of $T_{ITL^e}^\omega$

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Lemma

\mathcal{S}^ω is a countable set.

Strong Completeness of $T_{ITL^e}^\omega$

The sketch of proof

Definition

The pair (Γ, Δ) of formulas of \mathcal{L}_{ITL} is T_e^ω -saturated if

1. Γ is finitely closed under derivability, i.e. if $\Gamma' \subseteq \Gamma$ finite and $\Gamma' \vdash_{T_e^\omega} \varphi$, then $\varphi \in \Gamma$,
2. Γ has the disjunction property, i.e. if $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$,
3. Γ has the diamond property, i.e. if $\diamond\varphi \in \Gamma$ then $\bigcirc^n \varphi \in \Gamma$ for some $n \in \omega$,
4. Δ has the witness property, i.e. if $(\Sigma, \varphi) \in \mathcal{S}^\omega$ and $\varphi \in \Delta$, then there exists $\sigma \in \Sigma$ such that $\sigma \in \Delta$,
5. (Γ, Δ) is disjoint, i.e. $\Gamma \cap \Delta = \emptyset$ and $\perp \notin \Gamma$, and
6. (Γ, Δ) is full, i.e. $\Gamma \cup \Delta$ is the set of all formulas of \mathcal{L}_{ITL} .

Strong Completeness of $T_{ITL^e}^\omega$

The sketch of proof

Definition

Let Γ and Δ be two sets of formulas of \mathcal{L}_{ITL} . The pair (Γ, Δ) is called **T_e^ω -consistent** if $\Gamma \vdash_{T_e^\omega} \delta_1 \vee \dots \vee \delta_m$ holds for no $\delta_1, \dots, \delta_m \in \Delta$ and $m \in \omega$. In particular, this means that $\Gamma \not\vdash_{T_e^\omega} \perp$ since $\bigvee \emptyset = \perp$.

Strong Completeness of $T_{ITL^e}^\omega$

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Proposition

Let (Γ, Δ) be a T_e^ω -consistent pair where Δ is a finite set. Then, for any formula φ of \mathcal{L}_{ITL} , either $(\Gamma \cup \{\varphi\}, \Delta)$ or $(\Gamma, \Delta \cup \{\varphi\})$ is T_e^ω -consistent.

Strong Completeness of T_{ITL}^ω

The sketch of proof

Definition (Canonical Model)

The canonical model is the tuple $\mathfrak{M}_c = \langle W_c, \preceq_c, f_c, v_c \rangle$ where

- W_c is the set of all T_e^ω -saturated pairs;
- \preceq_c is the binary relation defined by $w \preceq_c w'$ iff $\Gamma \subseteq \Gamma'$ iff $\Delta' \subseteq \Delta$, for any $w = (\Gamma, \Delta)$ and $w' = (\Gamma', \Delta')$ in W_c ;
- f_c is the function defined by $f_c(w) = (\mathbf{f}_c(\Gamma), \mathbf{f}_c(\Delta))$ for any $w = (\Gamma, \Delta) \in W_c$ s.t. $\mathbf{f}_c(\Gamma) = \{\varphi \mid \bigcirc\varphi \in \Gamma\}$ and $\mathbf{f}_c(\Delta) = \{\varphi \mid \bigcirc\varphi \in \Delta\}$;
- v_c is the valuation defined by $v_c(p) = \{(\Gamma, \Delta) \in W_c \mid p \in \Gamma\}$ for any $p \in \mathbb{P}$.

Lemma

\mathfrak{M}_c is an intuitionistic Kripke model.

Strong Completeness of $T_{ITL^e}^\omega$

The sketch of proof

Existence Lemma

If $\Gamma \not\vdash_{T_e^\omega} \varphi$ then there exists a T_e^ω -saturated pair $w^* = (\Gamma^*, \Delta^*)$ such that $\Gamma \subseteq \Gamma^*$ and $\varphi \in \Delta^*$.

Strong Completeness of $T_{ITL_e}^\omega$

The sketch of proof

Main Lemma

Let $w = (\Gamma, \Delta)$ be a T_e^ω -saturated pair and $\psi \rightarrow \chi \in \Delta$. Then there exists a T_e^ω -saturated pair $w^* = (\Gamma^*, \Delta^*)$ such that $\Gamma \cup \{\psi\} \subseteq \Gamma^*$ and $\chi \in \Delta^*$.

Strong Completeness of $T_{ITL_e}^\omega$

The sketch of proof

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- Note that this lemma can be derived from Existence Lemma, if in a T_e^ω -saturated pair (Γ, Δ) , Γ is closed under derivability.

Strong Completeness of $T_{ITL_e}^\omega$

The sketch of proof

Main Lemma

Let $w = (\Gamma, \Delta)$ be a T_e^ω -saturated pair and $\psi \rightarrow \chi \in \Delta$. Then there exists a T_e^ω -saturated pair $w^* = (\Gamma^*, \Delta^*)$ such that $\Gamma \cup \{\psi\} \subseteq \Gamma^*$ and $\chi \in \Delta^*$.

- Note that this lemma can be derived from Existence Lemma, if in a T_e^ω -saturated pair (Γ, Δ) , Γ is closed under derivability.
- But, the derivation relation $\vdash_{T_e^\omega}$ is infinitary, so the fact that a set is finitely closed under derivability does not automatically implies that it is derivably closed.

Strong Completeness of $T_{ITL^e}^\omega$

The sketch of proof

Truth Lemma

Consider the above mentioned canonical model \mathfrak{M}_c . Then for any formula $\varphi \in \mathcal{L}_{ITL}$ and $w = (\Gamma, \Delta) \in W_c$, we have

$\varphi \in \Gamma$ implies $\mathfrak{M}_c, w \models \varphi$,

$\varphi \in \Delta$ implies $\mathfrak{M}_c, w \not\models \varphi$.

Strong Completeness of $T_{ITL^e}^\omega$

The sketch of proof

Theorem (Ch & Moniri 2021)

The proof system $T_{ITL^e}^\omega$ is strongly sound and complete for the class \mathcal{K} of all dynamic Kripke frames, i.e. $\Gamma \vdash_{T_{ITL^e}^\omega} \varphi$ iff $\Gamma \vDash_{\mathcal{K}} \varphi$.

Corollary

Let $w = (\Gamma, \Delta)$ be a $T_{ITL^e}^\omega$ -saturated pair. Then, Γ is closed under the derivation relation $\vdash_{T_{ITL^e}^\omega}$.

S. Chopoghloo and M. Moniri (2021). A strongly complete axiomatization of intuitionistic temporal logic. *Journal Logic and Computation*, 31(7), 1640–1659.

Some Results

Theorem

- The logic ITL^c (with continuous functions) is axiomatized by replacing the axiom scheme $\Box\varphi \rightarrow \varphi \wedge \bigcirc\Box\varphi$ in T_e^ω with $\Box\varphi \rightarrow \varphi \wedge \Box\bigcirc\varphi$.
- The logic ITL^p (persistent frames) is axiomatized by extending T_e^ω with the axiom scheme $(\bigcirc\varphi \rightarrow \bigcirc\psi) \rightarrow \bigcirc(\varphi \rightarrow \psi)$.

Gödel-Tarski Translation

Lemma

Let φ be a formula of \mathcal{L}_{ITL} and Γ be a set of formulas. Then,

$$\Gamma \vdash_{T_e^\omega} \varphi \text{ iff } t(\Gamma) \vdash_{T_{\mathcal{A}}^\omega} t(\varphi)$$

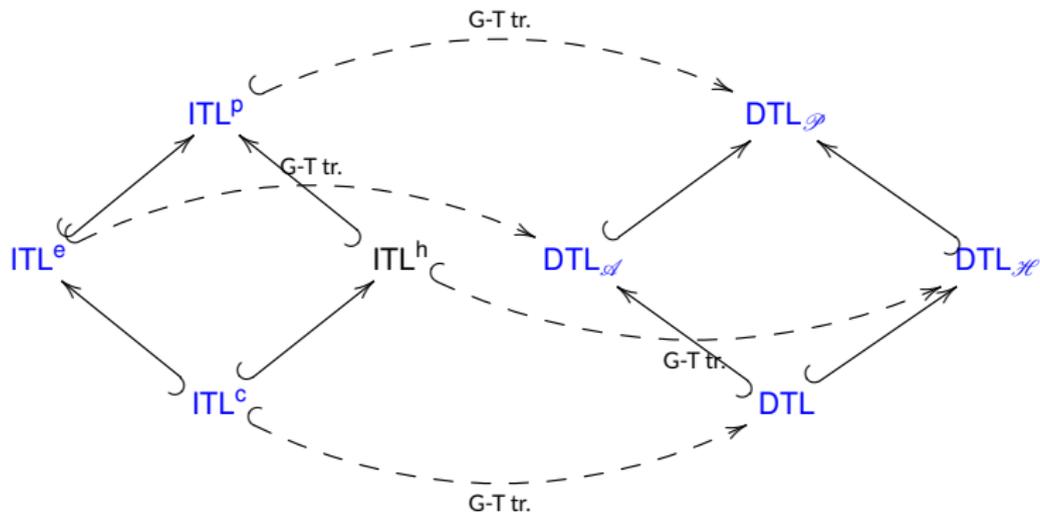
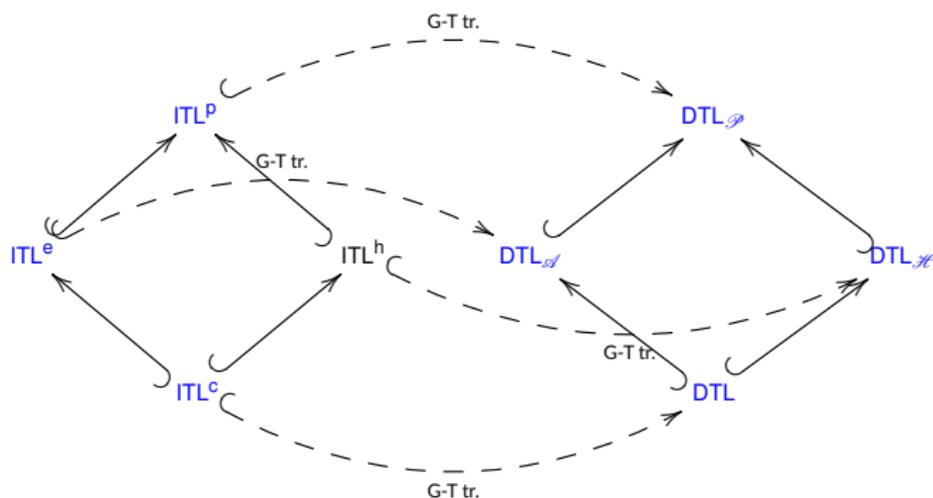


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Some Open Problems

The logics DTL , $DTL_{\mathcal{A}}$, ITL^c , and ITL^e are recursively enumerable. Hence, by [Craig's Theorem](#), each admits a [complete recursive axiomatization](#). However, finding such an axiomatization for these logics remains an open problem.



Recent Directions

- An alternative interpretation of the topological modality \blacklozenge , based on the [Cantor derivative operator](#), yields a version of DTL that is finitely axiomatizable, with its natural axiomatisation being both sound and complete over the class of scattered spaces.
- Extending ITL with the [co-implication connective](#) from Heyting-Brouwer logic yields a system known as [bi-intuitionistic linear temporal logic](#).

D. Fernández-Duque, and Y. Montacute (2023). Untangled: A complete dynamic topological logic. In *Proceedings of the AAI Conference on Artificial Intelligence (AAAI-23)*, 6355–6362, AAAI Press.

D. Fernández-Duque, B. McLean, and Z. Zenger (2024). A sound and complete axiomatisation for intuitionistic linear temporal logic. In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning (KRR-2024)*, 21(1), 350–360.

Thank you for your attention!